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Free boundary problem for a layer of inhomogeneous fluid

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Abstract

We consider a horizontal rectangular piece of layer of viscous inhomogeneous fluid, bounded below by a rigid plane and above by a free surface with lateral periodicity conditions. In the presence of gravity and surface tension, the fluid admits many nonhomogeneous rest states each of which corresponds to a given vertical density distribution. We study some stability properties of a rest state with plane free boundary Γ, corresponding to the given (large potential) gravitational force, given volume V, and given mass M, in the class of motions deriving by perturbations of initial data into the same volume V and having the same total mass M.

We assume the existence of global smooth nonsteady flows and study the control in time for the L^2 norms of perturbations to velocity, density, and height, in terms of their values at initial time. In the class of linear basic density profiles, we solve the problem of stability in the mean. The model does not admit a decay in time for the density, nevertheless, we prove a decay to zero for the L^2 norm of the velocity gradient and height gradient along a sequence of times. Furthermore, if we do not perturb the initial density then we can prove that more regular norms of perturbations to velocity, density, and height are bounded for all times. Finally, for the homogeneous basic density distribution, the asymptotic decay to zero of these more regular norms of perturbations to velocity, density, and height takes place.

1. Introduction

Let us consider a horizontal layer of viscous heavy fluid with variable density and free upper surface, and let us orient the vertical axis x_3 upward. As well known, in this case there exist numerous rest states with different density distributions. The character of this equilibrium strongly depends on the distribution of density through the layer. Actually, even though the pressure is always decreasing in x_3, it is possible to assume that the distribution of density is increasing with height, in this case the so-called Rayleigh–Taylor instability arises. The problem has been studied from linear point of view in [1,2]. Recently, Padula and Solonnikov [3] proved nonlinear exponential decay of the perturbations to the rest state for the case when the layer is filled by incompressible homogeneous fluid or by isothermal fluid in the presence of gravitational force only (see also [4]).

Nonlinear stability of rest for incompressible heterogeneous fluids remains still an open problem, and in this note we study this case. Our point of view is that the model of inhomogeneous fluids has physical draws back. Indeed, we point out several contradictory aspects and mathematical difficulties arising in this problem, and give a physical explanation of them. At first, we prove that the rest state admits only flat free surfaces, then we exhibit different rest states (with different pressures and densities profiles) in a fixed cell of layer, having given volume V, and filled by inhomogeneous fluid with the given mass M. A priori, there is no way to prefer one to another, even in the class of densities and pressures decreasing with height. A naive picture of such rest states is a stratification of several (horizontal) plane layers each of which is constituted by a homogeneous

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fluid: they satisfy the same boundary and side conditions, and therefore constitute an example of nonuniqueness in the class of steady flows. As a consequence of nonuniqueness of a rest state, there is no hope to obtain asymptotic stability of any basic rest state. We consider the evolution problem arising when we perturb the rest state. Notice that any initial density field ρ_0 may be considered as perturbation to a linear decreasing basic density field. Therefore, it is always possible to fix as basic density distribution a decreasing linear function of the vertical variable x_3. In this note, we provide a control in time for the L^2 norms of perturbations to velocity, density, and height by their values at initial time.

Precisely, we set free boundary problem for a inhomogeneous incompressible fluid in a way similar as that proposed in [3], and provide a control of L^2-norms of perturbations to the variables of the motion for all time instant (stability in the mean). Also, we prove decay to zero for the L^2 norm of the velocity gradient and height gradient along a sequence of times. Furthermore, for the sufficiently small perturbations to initial density, we show that more regular norms of perturbations to velocity, density, and height are bounded for all times.

Notice that, in a horizontal layer of inhomogeneous heavy fluid it is physically expected decay to rest for perturbations to velocity and height. Here, we first prove decay to zero of the L^2 norm of velocity through a sequence of times. Then we deduce decay to zero of perturbations to the height by noticing that the limit motion is again the rest.

Our uniqueness and stability method employs the ideas developed in [3,5,6] where a generalized energy functional is introduced. However, in this case, the equation of conservation of mass can no more be combined with dissipative effects present into the momentum equation, because the pressure is an independent term, and the density is constant through material lines. This can be a physical explanation of the fact that in the energy inequality does not appear any dissipative term corresponding to variation of density.

We are not going to present here a global in time existence result, in this connection, we mention that a local in time existence theorem for the problem under consideration is now in preparation.

The motion of incompressible fluids with varying densities in domains with fixed boundaries in the case of small external forces is considered in [7–14], e.g., We remark also the results of Simon, [12], concerning existence of solutions in bounded domains with rigid connected boundaries in the case of large external forces $\mathbf{f}$, summable in time. To our knowledge, in the case of constant external forces (like the gravity!) stability in the mean of the rest for inhomogeneous incompressible fluids, say control of a solution in a weak norm by initial data, for all $t \in (0, \infty)$, is not yet known.

2. The rest state

We consider a horizontal layer of incompressible inhomogeneous fluid bounded below by a rigid plane and assume that the upper surface is free. Let $x_1,\ x_2$ be the axes on the rigid plane and x_3 be the axis towards the upper surface Γ. In this case the gravitational force can be written in the form $\mathbf{f} = -g\nabla x_3$. We consider the periodic problem with respect to the variables $x' = (x_1, x_2)$, denote the periodicity cell by Σ, and assume that the free surface Γ can be posed by the equation $x_3 = \zeta(x')$. At first we consider the rest state with zero velocity and plane free boundary for the following problem

$$\begin{aligned}
&\rho\mathbf{u}\cdot\nabla\mathbf{u} - \mu\Delta\mathbf{u} = -\nabla p - \rho g\nabla x_3 \quad \text{in } \Omega = \left\{(x', x_3) \mid x' \in \Sigma,\ 0 < x_3 < \zeta(x')\right\},\\
&\nabla\cdot\mathbf{u} = 0,\ x \in \Omega,\\
&\nabla\cdot(\rho\mathbf{u}) = 0,\ x \in \Omega,\\
&\mathbf{u}\cdot\mathbf{n}\,|_\Gamma = 0,\\
&\mathbf{T}(x', \zeta)\mathbf{n}\,|_\Gamma = (\alpha\mathcal{H} - p_e)\mathbf{n}\,|_\Gamma,\\
&\mathbf{u}(x', 0) = 0,\\
&\int_\Omega \rho(x)\,\mathrm{d}x = M, \quad |\Omega| = V,
\end{aligned} \tag{2.1}$$

where $\mathbf{T} = -p\mathbf{I} + 2\mu\mathbf{D}$ is the stress tensor,

$$\mathbf{D} = \frac{\nabla\mathbf{u} + \nabla\mathbf{u}^{\mathrm{T}}}{2},$$

$\mathbf{I}$ is the unit matrix, μ and α are positive constants shear viscosity and surface tension respectively, $\mathbf{n}\,|_\Gamma = \mathbf{n}(x', \zeta(x'))$ is the unit outward normal on the free boundary, $\mathcal{H}$ is the double mean curvature of Γ which is negative for convex domains and is given by the formula

$$\mathcal{H} = \nabla'\cdot\left(\frac{\nabla'\zeta}{\sqrt{1 + |\nabla'\zeta|^2}}\right),$$

here ∇' is the gradient with respect to x'. The positive constants g and p_e are the gravity constant and the external pressure respectively.

It is easy to check that $\mathbf{u}=0$, $\rho_b=\rho_b(x_3)$, $\zeta_b=h=V/|\Sigma|$, $p_b(x_3)=g\int_{x_3}^h \rho_b(s)\,\mathrm{d}s+p_b(h)$ with $p_b(h)=p_e$ is a particular solution of system (2.1). We shall call this solution the basic flow.

By $W_\natural^{k,2}(\Omega)$ we denote a Sobolev function space with the norm

$$\|f\|_{W_\natural^{k,2}(\Omega)}=\Bigl(\sum_{|j|\leqslant k}\int_\Omega \bigl|D^j f(x)\bigr|^2\,\mathrm{d}x\Bigr)^{1/2},$$

symbol $\natural$ means periodicity. We introduce the regularity class

$$\mathcal{V}_s:=\bigl\{p(x',x_3),\mathbf{u}(x',x_3),\zeta(x'),\rho(x',x_3)\in C_\natural^1(\Omega)\times W_\natural^{1,2}(\Omega)\times W_\natural^{1,1}(\Sigma)\cap L_\natural^\infty(\Sigma)\times C_\natural(\Omega)\bigr\}.$$

We prove that in the class $\mathcal{V}_s$ problem (2.1) admits only basic rest states.

Theorem 1. *Any regular solution* $(p,\mathbf{u},\zeta,\rho)\in\mathcal{V}_s$ *of problem* (2.1) *satisfies the following relations*

$$\mathbf{u}=0,\quad p=p(x_3),\quad \rho=\rho(x_3).$$

Furthermore, the free boundary is the plane described by the equation $\zeta(x')=V/|\Sigma|=h$.

Proof. We multiply Eq. $(2.1)_1$ by $\mathbf{u}$, integrate over Ω, and arrive at

$$\int_\Omega \rho(\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\mathbf{u}\,\mathrm{d}x-\mu\int_\Omega \Delta\mathbf{u}\cdot\mathbf{u}\,\mathrm{d}x=-\int_\Omega \nabla p\cdot\mathbf{u}\,\mathrm{d}x-\int_\Omega \rho g\nabla x_3\cdot\mathbf{u}\,\mathrm{d}x.$$

Taking into account that $\nabla\cdot\mathbf{u}=0$, $\nabla\cdot(\rho\mathbf{u})=0$, we integrate by parts and obtain

$$\frac12\int_\Gamma \rho|\mathbf{u}|^2\mathbf{u}\cdot\mathbf{n}\,\mathrm{d}s+2\mu\int_\Omega \mathbf{D}:\mathbf{D}\,\mathrm{d}x-\int_\Gamma \mathbf{u}\cdot\mathbf{Tn}\,\mathrm{d}s=g\int_\Gamma \zeta\rho\mathbf{u}\cdot\mathbf{n}\,\mathrm{d}s. \tag{2.2}$$

Together with boundary conditions $(2.1)_4$, $(2.1)_5$ relation (2.2) implies $\int_\Omega \mathbf{D}:\mathbf{D}=0$, that, by Korn inequality (see, for example, [14]) leads to the identity $\mathbf{u}=0$.

Substituting $\mathbf{u}=0$ in problem (2.1), we have

$$\begin{aligned}&-\nabla p=\rho g\nabla x_3,\quad x\in\Omega,\\ &-(p(\zeta)-p_e)=\alpha\nabla'\cdot\frac{\nabla'\zeta}{\sqrt{1+|\nabla'\zeta|^2}},\quad x\in\Gamma.\end{aligned}\tag{2.3}$$

Eq. $(2.3)_1$ implies that p and ρ are independent on x_1, x_2, and the pressure is a decreasing function of x_3.

We set $\zeta=h+\eta$. By conservation of volume we have

$$\int_\Sigma \eta(x')\,\mathrm{d}x'=0.$$

We multiply $(2.3)_2$ by η and integrate over Σ, it gives us the relation

$$-\int_\Sigma \bigl(p(\zeta)-p(h)\bigr)\eta\,\mathrm{d}x'=-\int_\Sigma \bigl(p(\zeta)-p_e\bigr)\eta\,\mathrm{d}x'=-\alpha\int_\Sigma \frac{|\nabla'\eta|^2}{\sqrt{1+|\nabla'\eta|^2}}\,\mathrm{d}x'. \tag{2.4}$$

From Lagrange mean value theorem we have $(p(\zeta)-p(h))\eta=\mathrm{d}p/\mathrm{d}x_3\,|_{\bar\zeta}\,\eta^2\leqslant 0$, because the pressure is decreasing in x_3 for any density distribution. Since the left-hand side of (2.4) is nonnegative and the right-hand side of (2.4) is nonpositive, both sides of (2.4) are equal to zero. Consequently, $\nabla'\eta=0$, by conservation of volume it implies $\eta=0$.

Let us consider the question of uniqueness of the rest state (with zero velocity and plane free boundary) corresponding to the given periodicity cell Σ, the given total volume V, the given total mass M, and the given external pressure p_e. Let $\rho_{b1}(x_3)$ be a density of the rest state. Assume that there exists another rest state with the same Σ, V, M, p_e and denote it's density by $\rho_b(x_3)=\rho_{b1}(x_3)+\sigma(x_3)$. As the rest state occupies the domain $\Sigma\times[0,h]$, where $h=V/|\Sigma|$, conservation of mass gives us the following relation:

$$\int_\Sigma\int_0^h (\rho_{b1}+\sigma)\,\mathrm{d}x_3\,\mathrm{d}x'=\int_\Sigma\int_0^h \rho_b\,\mathrm{d}x_3\,\mathrm{d}x'.$$

It implies

$$\int_0^h \sigma(x_3)\,\mathrm{d}x_3 = 0. \tag{2.5}$$

Consequently, there are different rest states with zero velocity in the same domain $\Sigma \times [0,h]$ corresponding to the same total mass M and the same external pressure p_e with densities from the class

$$R = \left\{ \rho_b(x_3) = \rho_{b1}(x_3) + \sigma(x_3),\ \int_0^h \sigma(x_3)\,\mathrm{d}x_3 = 0 \right\}.$$

For any density $\rho_b \in \mathbb{R}$ the related pressure is uniquely determined from the following Cauchy problem

$$\frac{\mathrm{d}p_b}{\mathrm{d}x_3} = -g\rho_b, \qquad p_b(h) = p_e,$$

and hence is equal to

$$p_b(x_3) = g\int_{x_3}^h \rho_b(s)\,\mathrm{d}s + p_e.$$

We can identify the rest state only in case when we have additional information about the functions $\rho(x_3)$ and $p(x_3)$. For example, the rest state is unique in the class of regular solutions of problem (2.1) with pressures which differ from each other by monotone functions, namely, the following uniqueness theorem takes place. □

Theorem 2. *Let are given the values Σ, V, M, p_e, the assumptions of Theorem* 1 *hold true, and $S_b = (p_b, 0, h, \rho_b) \in \mathcal{V}_s$ is a solution of problem* (2.1)*. Then S_b is a unique solution of problem* (2.1) *in the class*

$$\mathcal{V}_{sp_b} = \left\{ (p, \mathbf{u}, \zeta, \rho) \in \mathcal{V}_s,\ p - p_b \text{ is a monotone function} \right\}.$$

Proof. Assume by absurdum that there is another solution of problem (2.1): $(p, \mathbf{u}, \zeta, \rho) \in \mathcal{V}_{sp_b}$. By Theorem 1

$$\mathbf{u} = 0, \quad \zeta = h, \quad \rho = \rho(x_3), \quad p = p(x_3).$$

We introduce the notations

$$\sigma(x_3) = \rho(x_3) - \rho_b(x_3), \qquad \pi(x_3) = p(x_3) - p_b(x_3).$$

As a consequence of (2.3), we have

$$-\frac{\mathrm{d}}{\mathrm{d}x_3}\pi(x_3) = g\sigma(x_3), \quad 0 < x_3 < h, \qquad \pi(h) = 0. \tag{2.6}$$

Under the assumption that $p - p_b$ is a monotone function, we have either $\mathrm{d}\pi/\mathrm{d}x_3 \geqslant 0$ or $\mathrm{d}\pi/\mathrm{d}x_3 \leqslant 0$, hence, from (2.6) it follows that the perturbation of density has a definite sign: $\sigma(x_3) \leqslant 0$ for $x_3 \in (0,h)$ or $\sigma(x_3) \geqslant 0$ for $x_3 \in (0,h)$. From (2.5) we deduce that $\sigma(x_3) = 0$ for $x_3 \in [0,h]$. Substituting $\sigma = 0$ in (2.6), we obtain $\pi = 0$, consequently, $\rho = \rho_b$, $p - p_b$. □

Now we present another uniqueness theorem, assuming that the density depends linearly on the variable x_3, in this case the density of the rest state can be uniquely determined.

Theorem 3. *Let are given the total mass M, the periodicity cell Σ, the volume V, the value of external pressure p_e, and the value of the density on the free boundary: $\rho|_{x \in \Gamma} = A$, $A \in (0, M/V)$. Then there exists a rest state S_b with plane free boundary Γ posed by the equation $x_3 = \zeta_b = V/|\Sigma| = h$, the density $\rho_b = c_1 x_3 + c_2$, where*

$$c_1 = 2\left(\frac{A|\Sigma|}{V} - \frac{M|\Sigma|}{V^2}\right), \qquad c_2 = 2\frac{M}{V} - A,$$

and the pressure

$$p_b = p_e - \frac{gc_1}{2}\left(x_3^2 - h^2\right) - gc_2(x_3 - h).$$

The rest state S_b is the unique solution of problem (2.1) *in the class of regular solutions with densities linearly depending on x_3 and having the same value A on Γ.*

Proof. By Theorem 1 we have $\mathbf{u}=0,\ \rho=\rho(x_3),\ \zeta=h=V/|\Sigma|$. Let $\rho_b=c_1x_3+c_2,\ c_1\neq 0$. Under our assumptions it holds the relation $(c_1x_3+c_2)|_\Gamma=A$, taking into account conservation of mass, we have for the coefficients c_1 and c_2 the following system:

$$\begin{cases} c_1h+c_2=A, \\ c_1\dfrac{h^2}{2}+c_2h=\dfrac{M}{|\Sigma|}. \end{cases}$$

This system has a unique solution

$$c_1=2\left(\frac{A}{h}-\frac{M}{|\Sigma|h^2}\right)=2\left(\frac{A|\Sigma|}{V}-\frac{M|\Sigma|}{V^2}\right),\qquad c_2=\frac{2M}{|\Sigma|h}-A=2\frac{M}{V}-A.$$

Note, that under the assumption $A\in(0,M/V)$, the density ρ_b satisfies the natural conditions $\rho_b(0)>\rho_b(h)>0$. We find the pressure p_b from the Cauchy problem

$$\frac{\mathrm{d}p_b}{\mathrm{d}x_3}=-g(c_1x_3+c_2),\qquad p_b(h)=p_e,$$

which has the following solution

$$p_b(x_3)=p_e-g\frac{c_1}{2}\big(x_3^2-h^2\big)-gc_2(x_3-h).\qquad\square$$

3. Stability of rest state

Now we consider the nonsteady free boundary problem in the periodic domain lying between the plane $x_3=0$ and the free surface Γ_t which can be given by the equation $x_3=\zeta(x',t)$. For inhomogeneous incompressible flows the problem is to find the domain $\Omega_t=\{(x',x_3)\mid x'\in\Sigma,\ 0<x_3<\zeta(x',t)\}$ filled by the fluid, the velocity vector field $\mathbf{u}$, the density ρ, and the pressure p – solution to the following problem

$$\begin{aligned}
&\rho\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t}-\mu\Delta\mathbf{u}=-\nabla p-\rho g\nabla x_3,\quad x\in\Omega_t,\\
&\nabla\cdot\mathbf{u}=0,\quad x\in\Omega_t,\\
&\frac{\mathrm{d}\rho}{\mathrm{d}t}=0,\quad x\in\Omega_t,\\
&\mathbf{u}(x',0,t)=0,\quad x'\in\Sigma,\\
&\mathbf{Tn}=(\alpha\mathcal{H}-p_e)\mathbf{n},\quad x\in\Gamma_t,\\
&W_n=\mathbf{u}\cdot\mathbf{n},\quad x\in\Gamma_t,\\
&\int_{\Omega_t}\rho(x,t)\,\mathrm{d}x=M,\quad |\Omega_t|=V,
\end{aligned}\tag{3.1}$$

where $\mathrm{d}\mathbf{u}/\mathrm{d}t=\partial\mathbf{u}/\partial t+(\mathbf{u}\cdot\nabla)\mathbf{u}$, by W_n we denote the velocity of the free boundary in the direction of the external normal $\mathbf{n}$. Under our assumptions $\mathbf{n}=(-\zeta_{x_1}/\sqrt{1+|\nabla'\zeta|^2},-\zeta_{x_2}/\sqrt{1+|\nabla'\zeta|^2},1/\sqrt{1+|\nabla'\zeta|^2})^{\mathrm{T}}$, hence condition $(3.1)_6$ can be written in the form

$$\frac{\partial\zeta}{\partial t}n_3=\mathbf{u}\cdot\mathbf{n},\quad x\in\Gamma_t.$$

Let us introduce the regularity class

$$\mathcal{V}:=\Big\{p(x,t),\mathbf{u}(x,t),\eta(x',t),\rho(x,t)\mid p\in C^1_\natural(\Omega_t),\ \mathbf{u}\in L^2\big(0,\infty;W^{1,2}_\natural(\Omega_t)\big)\cap L^\infty\big(0,\infty;L^3_\natural(\Omega_t)\big),$$
$$\eta\in L^\infty\big(0,T;W^{1,\infty}_\natural(\Sigma)\big),\ |\eta|<\frac{h}{2},\ \rho\in C_\natural(\Omega_t)\Big\}.$$

By '*regular solutions*' we denote solutions belonging to the class $\mathcal{V}$.

Theorem 4. *For any regular solution* $(p,\mathbf{u},h+\eta,\rho)$ *of problem* (3.1) *the integral identity*

$$\frac{\mathrm{d}}{\mathrm{d}t}\big[E_{\mathbf{u}}(t)+E_\eta(t)+gMx_{3G}(t)\big]+D_{\mathbf{u}}(t)=0\tag{3.2}$$

holds, where

$$E_{\mathbf{u}}(t)=\frac{1}{2}\int\limits_{\Omega_t}\rho|\mathbf{u}|^2\,\mathrm{d}x,\quad E_\eta(t)=\alpha\int\limits_{\Sigma}\bigl(\sqrt{1+|\nabla'\eta|^2}-1\bigr)\,\mathrm{d}x',\quad D_{\mathbf{u}}=2\mu\bigl\|\mathbf{D}(\mathbf{u})\bigr\|^2, \tag{3.3}$$

$x_{3G}(t)$ is the third coordinate of the center of mass.

Proof. We multiply Eq. $(3.1)_1$ by $\mathbf{u}$, integrate over Ω_t, and have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\Omega_t}\rho|\mathbf{u}|^2\,\mathrm{d}x=\int\limits_{\Omega_t}(\mu\Delta\mathbf{u}-\nabla p)\cdot\mathbf{u}\,\mathrm{d}x-\int\limits_{\Omega_t}g\rho\nabla x_3\cdot\mathbf{u}\,\mathrm{d}x,$$

then we integrate by parts and obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\Omega_t}\rho|\mathbf{u}|^2\,\mathrm{d}x+2\mu\int\limits_{\Omega_t}\mathbf{D}:\mathbf{D}\,\mathrm{d}x=\int\limits_{\Gamma_t}\mathbf{u}\cdot\mathbf{Tn}\,\mathrm{d}s-\int\limits_{\Omega_t}g\rho\nabla x_3\cdot\mathbf{u}\,\mathrm{d}x=I+II. \tag{3.4}$$

Boundary conditions imply

$$\begin{aligned}I&=\int\limits_{\Gamma_t}(\alpha\mathcal{H}-p_e)\mathbf{n}\cdot\mathbf{u}\,\mathrm{d}s=\alpha\int\limits_{\Sigma}\nabla'\cdot\frac{\nabla'\eta}{\sqrt{1+|\nabla'\eta|^2}}\eta_t\,\mathrm{d}x'-p_e\int\limits_{\Sigma}\eta_t\,\mathrm{d}x'=-\alpha\int\limits_{\Sigma}\frac{\nabla'\eta\cdot\nabla'\eta_t}{\sqrt{1+|\nabla'\eta|^2}}\,\mathrm{d}x'\\&=-\alpha\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\Sigma}\bigl(\sqrt{1+|\nabla'\eta|^2}-1\bigr)\,\mathrm{d}x',\end{aligned}$$

here we have used the conservation of volume, which gives us $\int_\Sigma\eta_t\,\mathrm{d}x'=0$. By virtue of $(3.1)_3$ and transport theorem, term II can be written in the form:

$$II=-g\int\limits_{\Omega_t}\rho\frac{\mathrm{d}x_3}{\mathrm{d}t}\,\mathrm{d}x=-g\int\limits_{\Omega_t}\frac{\mathrm{d}}{\mathrm{d}t}(\rho x_3)\,\mathrm{d}x=-g\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\Omega_t}\rho x_3\,\mathrm{d}x=-gM\frac{\mathrm{d}}{\mathrm{d}t}\bigl(x_{3G}(t)\bigr).$$

Substituting the obtained expressions into (3.4), we arrive at (3.2).

We integrate (3.2) in time and have

$$E_{\mathbf{u}}(t)+E_\eta(t)+gMx_{3G}(t)+\int\limits_0^t D_{\mathbf{u}}(\tau)\,\mathrm{d}\tau=E_{\mathbf{u}}(0)+E_\eta(0)+gMx_{3G}(0).\qquad\square \tag{3.5}$$

In the next theorem we control the perturbation to density, to this end we make the further assumption that the basic density $\rho_b(x_3)$ is a linear function.

Theorem 5. *Let $\mathbf{u}_b=0,\ \zeta_b=h,\ \rho_b(x_3)=ax_3+b,\ p_b(x_3)$ be the rest state corresponding to the given volume V and mass M. Assume that there exists a regular solution of problem* (3.1)*: $\mathbf{u},\ \rho=\rho_b+\sigma,\ \zeta=h+\eta,\ p=p_b+\pi$ in a domain having the same volume V and for a fluid having the same total mass M. Then the following energy identity*

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigl[E_{\mathbf{u}}(t)+E_\eta(t)+E_{\eta1}(t)+E_\sigma(t)\bigr]+D_{\mathbf{u}}(t)=0 \tag{3.6}$$

holds, where

$$\begin{aligned}&E_{\mathbf{u}}(t)=\frac{1}{2}\int\limits_{\Omega_t}\rho|\mathbf{u}|^2\,\mathrm{d}x,\quad E_\sigma(t)=-\frac{g}{2a}\int\limits_{\Omega_t}\sigma^2\,\mathrm{d}x,\quad a<0,\\&E_\eta(t)=\alpha\int\limits_{\Sigma}\bigl(\sqrt{1+|\nabla'\eta|^2}-1\bigr)\,\mathrm{d}x',\quad E_{\eta1}(t)=\frac{g}{2}\int\limits_{\Sigma}\bigl(a\zeta^*(x',t)+b\bigr)\eta^2(x',t)\,\mathrm{d}x',\\&D_{\mathbf{u}}=2\mu\bigl\|\mathbf{D}(\mathbf{u})\bigr\|^2,\end{aligned} \tag{3.7}$$

ζ^* lies between h and $h+\eta$.

Proof. In the same way as in the proof of Theorem 4, we multiply Eq. $(3.1)_1$ by $\mathbf{u}$, integrate by parts, and use boundary conditions, we arrive at

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\Omega_t}\rho|\mathbf{u}|^2\,\mathrm{d}x+2\mu\int\limits_{\Omega_t}\mathbf{D}:\mathbf{D}\,\mathrm{d}x+\alpha\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\Sigma}\big(\sqrt{1+|\nabla'\eta|^2}-1\big)\,\mathrm{d}x'$$
$$=-\int\limits_{\Omega_t}g\rho_b\nabla x_3\cdot\mathbf{u}\,\mathrm{d}x-\int\limits_{\Omega_t}g\sigma\nabla x_3\cdot\mathbf{u}\,\mathrm{d}x=I+II. \tag{3.8}$$

Let $\rho_b=ax_3+b$ with $a<0$. From the continuity equation we get

$$\frac{\partial\sigma}{\partial t}+\mathbf{u}\cdot\nabla\sigma=\frac{\mathrm{d}\sigma}{\mathrm{d}t}=-au_3. \tag{3.9}$$

We multiply (3.9) by $-\sigma/a$, integrate over Ω_t, and obtain

$$\int\limits_{\Omega_t}u_3\sigma\,\mathrm{d}x=-\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\Omega_t}\frac{\sigma^2}{2a}\,\mathrm{d}x. \tag{3.10}$$

Hence,

$$II=-g\int\limits_{\Omega_t}\sigma\nabla x_3\cdot\mathbf{u}\,\mathrm{d}x=\frac{g}{2a}\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\Omega_t}\sigma^2\,\mathrm{d}x. \tag{3.11}$$

By virtue of $(3.1)_2$, $(3.1)_6$, term I can be written in the form

$$I=-g\int\limits_{\Omega_t}\rho_b\nabla x_3\cdot\mathbf{u}\,\mathrm{d}x=-g\int\limits_{\Omega_t}\nabla\Bigg(\int\limits_h^{x_3}\rho_b(z)\,\mathrm{d}z\Bigg)\cdot\mathbf{u}\,\mathrm{d}x$$
$$=-g\int\limits_{\Sigma}\bigg(a\frac{\zeta^2-h^2}{2}+b\eta\bigg)\eta_t\,\mathrm{d}x'=-g\int\limits_{\Sigma}\bigg(a\frac{1}{6}\frac{\partial\eta^3}{\partial t}+\frac{ah}{2}\frac{\partial\eta^2}{\partial t}+\frac{b}{2}\frac{\partial\eta^2}{\partial t}\bigg)\,\mathrm{d}x'$$
$$=-g\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\Sigma}\frac{1}{2}(a\zeta^*+b)\eta^2\,\mathrm{d}x', \tag{3.12}$$

where $\zeta^*=h+\eta/3$.

By virtue of (3.8), (3.11), (3.12), we get the integral identity

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\Omega_t}\rho|\mathbf{u}|^2\,\mathrm{d}x+\alpha\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\Sigma}\big(\sqrt{1+|\nabla'\eta|^2}-1\big)\,\mathrm{d}x'+\frac{g}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\Omega_t}\frac{\sigma^2}{|a|}\mathrm{d}x+\frac{g}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\Sigma}\rho_b(\zeta^*)\eta^2\,\mathrm{d}x'+2\mu\big\|D(\mathbf{u})\big\|^2=0,$$

which implies the desired energy identity (3.6). □

Corollary 1. *Identity* (3.6) *implies stability in the mean. It means that we have control of the energy $E=E_{\mathbf{u}}(t)+E_\eta(t)+E_{\eta 1}(t)+E_\sigma(t)$, representing a norm of perturbations. Namely, $E(t)\leqslant E(0)$ for any positive t. In consequence of stability in the mean and relation* (3.5)*, the term $\int_0^t D_{\mathbf{u}}(\tau)\,\mathrm{d}\tau$ is bounded for any positive t, and the L^2 norm of the velocity gradient tends to zero along a sequence of times. Consequently, along this sequence of times, the motion tends to the rest, and perturbations to the height tends to zero.*

4. Estimates for the first order derivatives

Let at the initial time $\eta(x',0)=\eta_0(x')$, $\mathbf{u}(x,0)=\mathbf{u}_0(x)$, $\rho^0=\rho(x,0)=\rho_b(x_3)+\sigma^0(x)$, where the initial perturbations of the rest state $\mathbf{u}_0$, η_0, σ^0 are sufficiently small. In this section we assume that there exists a solution of problem (3.1) with the following regularity

$$\eta\in C^1\big(0,\infty;W_\natural^{2,2}(\Omega_t)\big),\quad \eta_t\in C^1\big(0,\infty;W_\natural^{1,2}(\Omega_t)\big),\quad \mathbf{u}\in C^1\big(0,\infty;W_\natural^{3,2}(\Omega_t)\big),$$
$$\mathbf{u}_t\in C^1\big(0,\infty;W_\natural^{1,2}(\Omega_t)\big),\quad |\eta|<\frac{h}{2},\quad \rho\in C_\natural^1(\Omega_t),\quad p\in C_\natural^2(\Omega_t),$$

and try to see what additional information about the behaviour of the solution we can have. In the first part we compute an energy identity for L^2 norms of spatial and time derivatives of perturbations to the velocity and to the height (see lemma), in the second part we provide estimates for the above norms (see Theorem 6).

Lemma. *Let $\mathbf{u}_b = 0,\ \zeta_b = h,\ \rho_b(x_3),\ p_b(x_3)$ be the rest state with volume V and mass M. Assume that there exists a regular solution of problem* (3.1)*: $\mathbf{u},\ \rho = \rho_b + \sigma,\ \zeta = h + \eta,\ p = p_b + \pi$ in a domain having the same volume V and for a fluid having the same total mass M. Then the following energy identity*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E} + \mathcal{D} = P^{(x_1)} + P^{(x_2)} + P^{(t)} + N(\mathbf{u}, \eta) \tag{4.1}$$

holds, where $\mathcal{E} = \mathcal{E}_{\mathbf{u}} + \mathcal{E}_\eta,\ \mathcal{D} = \mathcal{D}_{\mathbf{u}} + \mathcal{D}_\eta$,

$$\begin{aligned}
&\mathcal{E}_{\mathbf{u}} = \frac{1}{2}\int_{\Omega_t} \rho\biggl(\sum_{i=1,2} |\mathbf{u}_{x_i}|^2 + |\mathbf{u}_t|^2\biggr)\,\mathrm{d}x + \gamma_0 \int_{\Omega_t} \rho\mathbf{u}_t\cdot\mathbf{V}_t\,\mathrm{d}x + \sum_{i=1,2}\gamma_i \int_{\Omega_t} \rho\mathbf{u}_{x_i}\cdot\mathbf{V}_{x_i}\,\mathrm{d}x,\\
&\mathcal{E}_\eta = \frac{\alpha}{2}\bigl(K^{(t)} + K^{(x_1)} + K^{(x_2)}\bigr),\\
&K^{(x_i)} = \int_\Sigma \frac{|\nabla'\eta_{x_i}|^2 + (\eta_{x_1}\eta_{x_i x_2} - \eta_{x_2}\eta_{x_i x_1})^2}{\sqrt{(1+|\nabla'\eta|^2)^3}}\,\mathrm{d}x', \quad i = 1, 2,\\
&K^{(t)} = \int_\Sigma \frac{|\nabla'\eta_t|^2 + (\eta_{x_1}\eta_{t x_2} - \eta_{x_2}\eta_{t x_1})^2}{\sqrt{(1+|\nabla'\eta|^2)^3}}\,\mathrm{d}x',
\end{aligned} \tag{4.2}$$

$\gamma_0,\ \gamma_1,\ \gamma_2$ are small positive numbers, $\mathbf{V} \in L^\infty(0,\infty; W^{1,2}_\natural(\Omega_t))$ with $\partial\mathbf{V}/\partial t \in L^\infty(0,\infty; L^2_\natural(\Omega))$ is an auxiliary vector field constructed in [3]*, which satisfies the following conditions*

$$\begin{aligned}
&\nabla\cdot\mathbf{V} = 0, \quad x \in \Omega_t,\\
&\mathbf{V}(x', 0, t) = 0,\\
&\mathbf{V}\cdot\mathbf{n}|_{\Gamma_t} = \eta n_3,
\end{aligned}$$

and the following estimates

$$\begin{aligned}
&\left\|\frac{\partial\mathbf{V}}{\partial t}\right\|_{L_2(\Omega)} \leqslant c\left\|\frac{\partial\eta}{\partial t}\right\|_{L_2(\Sigma)}, \qquad \|\nabla\mathbf{V}\|_{L_2(\Omega)} \leqslant c\|\eta\|_{W^{1,2}(\Sigma)};\\
&\mathcal{D}_{\mathbf{u}} = 2\mu\bigl(\bigl\|\mathbf{D}(\mathbf{u}_t)\bigr\|^2 + \bigl\|\mathbf{D}(\mathbf{u}_{x_1})\bigr\|^2 + \bigl\|\mathbf{D}(\mathbf{u}_{x_2})\bigr\|^2\bigr),\\
&\mathcal{D}_\eta = \alpha\bigl(\gamma_1 K^{(x_1)} + \gamma_2 K^{(x_2)} + \gamma_0 K^{(t)}\bigr),\\
&P^{(x_i)} = -\int_{\Omega_t} \sigma_{x_i}\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t}\cdot(\mathbf{u}_{x_i} + \gamma_i\mathbf{V}_{x_i})\,\mathrm{d}x - g\int_{\Omega_t} \sigma_{x_i}\nabla x_3\cdot(\mathbf{u}_{x_i} + \gamma_i\mathbf{V}_{x_i})\,\mathrm{d}x, \quad i = 1, 2,\\
&P^{(t)} = -\int_{\Omega_t} \sigma_t\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t}\cdot(\mathbf{u}_t + \gamma_0\mathbf{V}_t)\,\mathrm{d}x - g\int_{\Omega_t} \sigma_t\nabla x_3\cdot(\mathbf{u}_t + \gamma_0\mathbf{V}_t)\,\mathrm{d}x,
\end{aligned}$$

$N(\mathbf{u}, \eta)$ contains nonlinear terms which do not depend on σ and are given in the proof.

We give the proof of lemma in Appendix.

Note that under the assumption that the density ρ is uniformly bounded in time: $C_1 \leqslant \rho \leqslant C_2,\ C_2 > C_1 > 0$, the norms $\|\mathbf{u}\|_{L_2(\Omega_t)},\ \|\sqrt{\rho}\mathbf{u}\|_{L_2(\Omega_t)},\ \|\rho\mathbf{u}\|_{L_2(\Omega_t)}$ are equivalent to each other. The generalized energy $\mathcal{E} = \mathcal{E}_{\mathbf{u}} + \mathcal{E}_\eta$ becomes equivalent to $\|\mathbf{u}_{x_i}\|_{L_2(\Omega_t)} + \|\mathbf{u}_t\|_{L_2(\Omega_t)} + \|\nabla\eta_t\|_{L_2(\Sigma)} + \|\nabla\eta_{x_i}\|_{L_2(\Sigma)}$ for the sufficiently small positive $\gamma_0, \gamma_1, \gamma_2$. In what follows we will sometimes use the designation $\|\mathbf{u}\|$ for the norm $\|\mathbf{u}\|_{L_2(\Omega_t)}$. By virtue of the relation $\int_\Sigma \eta(x', t)\,\mathrm{d}x' = 0$ and boundary condition $(3.1)_4$, L_2 norms of η, $\mathbf{u}$ can be estimated with the help of Korn inequality and embedding theorems by the norms of the derivatives. To simplify the estimates of the nonlinear boundary terms, we assume that

$$\alpha|\nabla'\eta_{x_i}|, \quad g|\nabla\rho|, \quad 2\mu\bigl|\mathbf{D}(\mathbf{u}_t)\bigr|, \quad |p_{x_3}|, \quad 2\mu\bigl|\mathbf{D}(\mathbf{u}_{x_j})\bigr| < \beta, \quad i = 1, 2,\ j = 1, 2, 3, \tag{4.3}$$

where β is a small positive number.

It should be mentioned that the part of restrictions (4.3) is not necessary, it is possible to estimate nonlinear terms without so strong restrictions, but, as our result cannot be improved in this way, we have all the assumptions (4.3) to avoid long calculations, which are presented in [4].

Theorem 6. *Let the initial data in problem* (3.1) *satisfy the following regularity conditions*:

$$\eta_0 \in W^{2,2}_{\natural}(\Sigma), \quad \rho^0 \in C^1_{\natural}(\Omega_0), \quad \mathbf{u}_0 \in W^{3,2}_{\natural}(\Omega_0), \quad \mathbf{u}^0_t \in W^{1,2}_{\natural}(\Omega_0),$$

the natural compatibility conditions, and the following smallness conditions

$$\mathcal{E}(\mathbf{u}_0, \eta_0) = \mathcal{E}(0) \leqslant \varepsilon, \qquad \left\|\nabla \rho^0\right\|_{L_2(\Omega)} \leqslant \mathcal{E}(0)\delta, \tag{4.4}$$

with a sufficiently small ε and with some small positive δ. Assume that there exists a global regular solution of problem (3.1), *satisfying conditions* (4.3) *with a small positive β (for example, we can take $\beta = 1/64$), then*

$$\mathcal{E}(t) \leqslant \mathcal{E}(0)$$

for any $t \in [0, T]$, where $T > 0$ depends on the initial perturbation of density $\|\nabla \rho^0\|_{L_2(\Omega)}$ and tends to infinity as this perturbation tends to zero.

Proof. Let us deduce the energy inequality from the identity (4.1). Nonlinear terms in $N(\mathbf{u}, \eta)$ can be estimated with the help of Holder inequality and embedding theorems in the same way as it is done, for example, in [4]. Precisely,

$$\left|\int_{\Omega_t} \rho(\mathbf{u}_{x_i} \cdot \nabla)\mathbf{u} \cdot (\mathbf{u}_{x_i} + \gamma_i \mathbf{V}_{x_i})\,\mathrm{d}x\right| \leqslant c|\mathbf{u}_{x_i}|_{L_\infty} \|\nabla \mathbf{u}\|_{L_2(\Omega_t)} \big(\|\mathbf{u}_{x_i}\|_{L_2(\Omega_t)} + \gamma_i \|\mathbf{V}_{x_i}\|_{L_2(\Omega_t)}\big) \leqslant c\mathcal{D}\sqrt{\mathcal{E}}.$$

Because of the fact that

$$\mathbf{n}^{\wedge}_{x_i} = (-\eta_{x_1 x_i}, -\eta_{x_2 x_i}, 0)^{\mathrm{T}}, \qquad \mathbf{n}^{\wedge}_t = (-\eta_{x_1 t}, -\eta_{x_2 t}, 0)^{\mathrm{T}},$$

any boundary term in $N(\mathbf{u}, \eta)$ contains at least three multipliers and under the assumption (4.3) can be estimated by $\beta\mathcal{D}$, or $c\mathcal{D}\sqrt{\mathcal{E}}$, By virtue of $(3.1)_3$, the second term in $P^{(t)}$ can be written in the form $g\int_{\Omega_t} \mathbf{u} \cdot \nabla \rho(\mathbf{u}_t + \gamma_0 \mathbf{V}_t) \cdot \nabla x_3 \,\mathrm{d}x$ and, under the assumption (4.3), estimated by $c\beta\mathcal{E}$.

The main difficulty is in the estimation of the terms $P^{(x_i)}$. As a consequence of the equation $\nabla' \rho_b = 0$, we have $\partial\sigma/\partial x_i = \partial\rho/\partial x_i$. Let us introduce the Lagrangian coordinates ξ connected with the vector field $\mathbf{u}$: $x = \xi + \int_0^t \mathbf{u}(\xi, \tau)\,\mathrm{d}\tau$. As it is proved in [9], under our regularity assumptions, for given $\mathbf{u}$ and ρ^0, the problem

$$\frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \nabla \rho = 0, \qquad \rho(x, t)|_{t=0} = \rho^0(x)$$

has a unique solution

$$\rho(x, t) = \rho^0\big(\xi(x, t)\big),$$

and the estimate

$$\left|\nabla_x \rho(x, t)\right| \leqslant c\left|\nabla_\xi \rho^0\right|_{\xi(x,t)}\left|\exp\left(\int_0^t |\nabla \mathbf{u}|_\infty(\tau)\,\mathrm{d}\tau\right)\right.$$

holds true. Hence, by the help of Hölder inequality, we obtain

$$\begin{aligned}\left|\int_{\Omega_t} \sigma_{x_i} \nabla x_3 \cdot \mathbf{u}_{x_i}\,\mathrm{d}x\right| &\leqslant \int_{\Omega_t} \left|\rho_{x_i}(x, t)\right| \left|\nabla x_3 \cdot \mathbf{u}_{x_i}\right| \mathrm{d}x \leqslant c\int_{\Omega_t} \left|\nabla_\xi \rho^0\big(\xi(x, t)\big)\right| \exp\left(\int_0^t |\nabla \mathbf{u}|_\infty(\tau)\,\mathrm{d}\tau\right) |\nabla x_3 \cdot \mathbf{u}_{x_i}|\,\mathrm{d}x \\ &\leqslant c\exp\left(\int_0^t |\nabla \mathbf{u}|_\infty(\tau)\,\mathrm{d}\tau\right) \left\|\nabla \rho^0\right\|_{L_2(\Omega)} \|\mathbf{u}_{x_i}\|_{L_2(\Omega_t)} \leqslant c\exp\left(\int_0^t |\nabla \mathbf{u}|_\infty(\tau)\,\mathrm{d}\tau\right) \left\|\nabla \rho^0\right\|_{L_2(\Omega)} \sqrt{\mathcal{E}}.\end{aligned} \tag{4.5}$$

It is clear that the other terms in $P^{(x_i)}$ are estimated in the same way.

We point out that because of the presence of the terms $P^{(x_i)}$, even under the very strong regularity and smallness assumptions, we are no more able to derive a priori estimates similar to those obtained in [4] for the case of compressible fluid. In our case, if we estimate all the terms at right-hand side in a way described above, for sufficiently small positive numbers γ_0, γ_1, γ_2, we arrive at the inequality

$$\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t}+\mathcal{D}\leqslant\frac{1}{2}\mathcal{D}+c\sqrt{\mathcal{E}}\left(\mathcal{D}+\exp\left(\int_0^t|\nabla\mathbf{u}|_\infty(\tau)\,\mathrm{d}\tau\right)\left\|\nabla\rho^0\right\|_{L_2(\Omega)}\right). \tag{4.6}$$

It is worth to remark that there is no estimate for the perturbation σ of density, and this will create the major difficulty in the sequel. Note also that, as we are going to derive the estimates under the smallness assumptions, we can be sure that at least $\mathcal{E}<1$, $\mathcal{D}<1$.

From the explicit formula (4.2), we can conclude that $\mathcal{E}<\nu\mathcal{D}$, $\nu>0$, and, under a suitable smallness assumption, derive from (4.6) the estimate

$$\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t}+A_1\mathcal{E}\leqslant A_2\exp\left(\int_0^t|\nabla\mathbf{u}|_\infty(\tau)\,\mathrm{d}\tau\right)\left\|\nabla\rho^0\right\|_{L_2(\Omega)}\sqrt{\mathcal{E}}, \tag{4.7}$$

where A_1, A_2 are positive constants. This estimate allows us to prove only the control of the energy $\mathcal{E}$ by the initial data on some finite interval of time.

We introduce the notation $y=\sqrt{\mathcal{E}}$. Under our regularity assumptions, $y(t)$ is a continuously differentiable function. Inequalities (4.4) and (4.7) imply

$$2\frac{\mathrm{d}y}{\mathrm{d}t}\leqslant A_2\exp\left(\int_0^t|\nabla\mathbf{u}|_\infty(\tau)\,\mathrm{d}\tau\right)\delta y(0)-A_1y(t).$$

Under our regularity assumptions, $|\nabla\mathbf{u}|$ is a bounded function, let

$$|\nabla\mathbf{u}|_\infty(\tau)|\leqslant A_3$$

for $\tau>0$. Let the number T satisfies the inequality

$$T<\frac{1}{A_3}\ln\frac{A_1}{A_2\delta}. \tag{4.8}$$

If we assume that

$$\max_{t\in[0,T]}y(t)=y(t^*), \tag{4.9}$$

where $t^*>0$, then for any $\tau\in[0,t^*]$ it holds the inequality $y(\tau)\leqslant y(t^*)$. Because of (4.8), (4.9), we have

$$A_2\exp\left(\int_0^{t^*}|\nabla\mathbf{u}|_\infty(\tau)\,\mathrm{d}\tau\right)\delta y(0)\leqslant A_2\,\mathrm{e}^{A_3T}\delta y(0)<A_2\frac{A_1}{A_2\delta}\delta y(0)\leqslant A_1y(t^*).$$

As a consequence,

$$\left.\frac{\mathrm{d}y}{\mathrm{d}t}\right|_{t=t^*}\leqslant\frac{1}{2}\left(A_2\exp\left(\int_0^{t^*}|\nabla\mathbf{u}|_\infty(\tau)\,\mathrm{d}\tau\right)\delta y(0)-A_1y(t^*)\right)<0.$$

It means that there is a left neighborhood of the point t^* where $y(t)>y(t^*)$ and we have contradiction with assumption (4.9). This allows us to conclude that

$$\max_{t\in[0,T]}y(t)=y(0),$$

which obviously implies the desired estimate

$$\mathcal{E}(t)\leqslant\mathcal{E}(0),\quad t\in[0,T],$$

where T satisfies restriction (4.8). $\square$

In the case when the rest state density is a linear function: $\rho_b = ax_3 + b$, we can take into account identity (3.6). Obviously, the term $\frac{\mathrm{d}}{\mathrm{d}t}E_\sigma$ can be written in the form $\int_{\Omega_t} \sigma \nabla x_3 \cdot \mathbf{u}\,\mathrm{d}x$. To estimate this term, we use condition $(3.1)_3$, which implies

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = -\mathbf{u}\cdot\nabla\rho_b = -au_3.$$

Consequently, in Lagrangian coordinates connected with the vector field $\mathbf{u}$, we have

$$\sigma(\xi,t) = \sigma(\xi,0) - a\int_0^t u_3(\xi,\tau)\,\mathrm{d}\tau,$$

and

$$|\sigma|_{\infty,\Omega_t} \leqslant |\sigma_0|_{\infty,\Omega_0} + |a|\int_0^t |u_3|_{\infty,\Omega_\tau}\,\mathrm{d}\tau.$$

Thus, we add (3.6) to (4.7), and have

$$\frac{\mathrm{d}\mathcal{E}^*}{\mathrm{d}t} + A_1^*\mathcal{E}^* \leqslant A_2^*\left(\exp\left(\int_0^t |\nabla\mathbf{u}|_\infty(\tau)\,\mathrm{d}\tau\right)\left\|\nabla\rho^0\right\|_{L_2(\Omega)} + \left|\sigma^0\right|_\infty\right)\sqrt{\mathcal{E}}, \tag{4.10}$$

where

$$\mathcal{E}^* = \mathcal{E} + E_\eta + E_{\eta 1}.$$

It is clear that on the base of inequality (4.10), in the same way as Theorem 6, the following result can be proved.

Theorem 7. *Let $\rho_b = ax_3 + b$, all the assumptions of Theorem* 6 *hold true, and, moreover, the following estimate takes place*

$$\left\|\nabla\rho^0\right\|_{L_2(\Omega)} + \left|\sigma^0\right|_\infty \leqslant \mathcal{E}^*(0)\delta,$$

where $\delta > 0$ is a sufficiently small number.

Then, under assumption on existence of a global regular solution to problem (3.1)*, we have*

$$\mathcal{E}^*(t) \leqslant \mathcal{E}^*(0)$$

for any $t \in [0,T]$, where $T > 0$ depends on the initial perturbation of density $\|\nabla\rho^0\|_{L_2(\Omega)} + |\sigma^0|_\infty$ and tends to infinity as this perturbation tends to zero.

Remark. Let us stress the fact that if the basic stratification is homogeneous (the density is equal to a constant), and we perturb the density only on the horizontal directions, we have proved decay to the rest only for the velocity and the height. The density distribution might not decay to the basic uniform one!

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Appendix

Proof of lemma. We take derivatives with respect to x_i, $i = 1, 2$, of all the equations in problem (3.1)

$$\begin{aligned}
&\sigma_{x_i}\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} + \rho\left(\frac{\mathrm{d}\mathbf{u}_{x_i}}{\mathrm{d}t} + \mathbf{u}_{x_i}\cdot\nabla\mathbf{u}\right) - \mu\Delta\mathbf{u}_{x_i} = -\nabla p_{x_i} - \sigma_{x_i} g\nabla x_3, && x\in\Omega_t,\\
&\nabla\cdot\mathbf{u}_{x_i} = 0, && x\in\Omega_t,\\
&\frac{\mathrm{d}\sigma_{x_i}}{\mathrm{d}t} + \mathbf{u}_{x_i}\cdot\nabla(\rho_b+\sigma) = 0, && x\in\Omega_t.
\end{aligned} \tag{A.1}$$

We multiply $(\mathrm{A}.1)_1$ by $\mathbf{u}_{x_i}$, integrate over Ω_t, and arrive at

$$\int_{\Omega_t} \sigma_{x_i} \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} \cdot \mathbf{u}_{x_i}\,\mathrm{d}x + \int_{\Omega_t} \rho \frac{\mathrm{d}\mathbf{u}_{x_i}}{\mathrm{d}t} \cdot \mathbf{u}_{x_i}\,\mathrm{d}x + \int_{\Omega_t} \rho(\mathbf{u}_{x_i} \cdot \nabla)\mathbf{u} \cdot \mathbf{u}_{x_i}\,\mathrm{d}x - \mu \int_{\Omega_t} \Delta\mathbf{u}_{x_i} \cdot \mathbf{u}_{x_i}\,\mathrm{d}x$$
$$= -\int_{\Omega_t} \nabla p_{x_i} \cdot \mathbf{u}_{x_i}\,\mathrm{d}x - g\int_{\Omega_t} \sigma_{x_i} \nabla x_3 \cdot \mathbf{u}_{x_i}\,\mathrm{d}x. \tag{A.2}$$

We take derivatives with respect to x_i, $i = 1, 2$, of the boundary conditions in problem (3.1), taking into account that the free boundary Γ_t is posed by the equation $x_3 = \zeta(x_1, x_2, t)$. Condition $(3.1)_5$ implies

$$2\mu\mathbf{D}(\mathbf{u})\mathbf{n}_{x_i} + 2\mu\mathbf{D}(\mathbf{u}_{x_i})\mathbf{n} + 2\mu\mathbf{D}(\mathbf{u}_{x_3})\mathbf{n}\zeta_{x_i} = (\alpha\mathcal{H} - p_e + p)\mathbf{n}_{x_i} + \left(\alpha\frac{\partial\mathcal{H}}{\partial x_i} + p_{x_i} + p_{x_3}\zeta_{x_i}\right)\mathbf{n}. \tag{A.3}$$

We differentiate the identity $\|\mathbf{n}\|^2 = 1$ with respect to x_i and see that $\mathbf{n}_{x_i} \cdot \mathbf{n} = 0$, consequently, from $(3.1)_5$ we have $\mathbf{n}_{x_i} \cdot \mathbf{D}(\mathbf{u})\mathbf{n} = 0$. Multiplying (A.3) by the vector $\mathbf{n}$, we obtain

$$p_{x_i} = 2\mu\mathbf{n} \cdot \mathbf{D}(\mathbf{u}_{x_3})\mathbf{n}\zeta_{x_i} - p_{x_3}\zeta_{x_i} - \alpha\frac{\partial\mathcal{H}}{\partial x_i} + 2\mu\mathbf{n} \cdot \mathbf{D}(\mathbf{u}_{x_i})\mathbf{n}. \tag{A.4}$$

Because of $(\mathrm{A}.1)_3$, (A.4), with the help of integration by parts, we obtain the following relation:

$$\int_{\Omega_t} (\mu\Delta\mathbf{u}_{x_i} - \nabla p_{x_i}) \cdot \mathbf{u}_{x_i}\,\mathrm{d}x = -2\mu\int_{\Omega_t} \mathbf{D}(\mathbf{u}_{x_i}) : \mathbf{D}(\mathbf{u}_{x_i})\,\mathrm{d}x + \int_{\Gamma_t} \mathbf{u}_{x_i} \cdot \big(-p_{x_i}\mathbf{n} + 2\mu\mathbf{D}(\mathbf{u}_{x_i})\mathbf{n}\big)\,\mathrm{d}s$$
$$= -2\mu\big\|\mathbf{D}(\mathbf{u}_{x_i})\big\|^2 + B_1^{(i)} + B_2^{(i)},$$

where

$$B_1^{(i)} = \alpha\int_{\Gamma_t} \frac{\partial\mathcal{H}}{\partial x_i}\mathbf{n} \cdot \mathbf{u}_{x_i}\,\mathrm{d}s,$$
$$B_2^{(i)} = \int_{\Gamma_t} p_{x_3}\zeta_{x_i}\mathbf{n} \cdot \mathbf{u}_{x_i}\,\mathrm{d}s - 2\mu\int_{\Gamma_t} \mathbf{u}_{x_i} \cdot \mathbf{n}\mathbf{n} \cdot \mathbf{D}(\mathbf{u}_{x_3})\mathbf{n}\zeta_{x_i}\,\mathrm{d}s + 2\mu\int_{\Gamma_t} \big(\mathbf{u}_{x_i} - (\mathbf{n} \cdot \mathbf{u}_{x_i})\mathbf{n}\big) \cdot \mathbf{D}(\mathbf{u}_{x_i})\mathbf{n}\,\mathrm{d}s. \tag{A.5}$$

At first we consider the term $B_1^{(i)}$. We rewrite condition $(3.1)_6$ in the form $\eta_t = \mathbf{u} \cdot \mathbf{n}^\wedge$, where $\mathbf{n}^\wedge = \sqrt{1 + |\nabla'\eta|^2}\,\mathbf{n}$ and take derivatives with respect to x_i, we have

$$\eta_{tx_i} = \mathbf{u}_{x_i} \cdot \mathbf{n}^\wedge + \mathbf{u}_{x_3} \cdot \mathbf{n}^\wedge\eta_{x_i} + \mathbf{u} \cdot \mathbf{n}^\wedge_{x_i},$$

consequently, $B_1^{(i)}$ reads

$$B_1^{(i)} = \alpha\int_\Sigma \frac{\partial\mathcal{H}}{\partial x_i}\mathbf{n}^\wedge \cdot \mathbf{u}_{x_i}\,\mathrm{d}x' = \alpha\int_\Sigma \frac{\partial\mathcal{H}}{\partial x_i}\eta_{tx_i}\,\mathrm{d}x' - \alpha\int_\Sigma \frac{\partial\mathcal{H}}{\partial x_i}\mathbf{u} \cdot \mathbf{n}^\wedge_{x_i}\,\mathrm{d}x' - \alpha\int_\Sigma \frac{\partial\mathcal{H}}{\partial x_i}\mathbf{u}_{x_3} \cdot \mathbf{n}^\wedge\eta_{x_i}\,\mathrm{d}x' = B_{11}^{(i)} + B_{12}^{(i)} + B_{13}^{(i)}.$$

In the term $B_{11}^{(i)}$ we integrate by parts and use the periodicity condition, we have:

$$B_{11}^{(i)} = \alpha\int_\Sigma \left\{\nabla' \cdot \left(\frac{\nabla'\eta_{x_i}}{\sqrt{1 + |\nabla'\eta|^2}}\right)\eta_{tx_i} - \nabla' \cdot \left(\frac{\nabla'\eta}{\sqrt{(1 + |\nabla'\eta|^2)^3}}\nabla'\eta \cdot \nabla'\eta_{tx_i}\right)\eta_{tx_i}\right\}\mathrm{d}x'$$
$$= -\alpha\int_\Sigma \frac{\nabla'\eta_{x_i} \cdot \nabla'\eta_{tx_i}}{\sqrt{1 + |\nabla'\eta|^2}}\,\mathrm{d}x' + \alpha\int_\Sigma \frac{\nabla'\eta(\nabla'\eta \cdot \nabla'\eta_{x_i}) \cdot \nabla'\eta_{x_it}}{(\sqrt{1 + |\nabla'\eta|^2})^3}\,\mathrm{d}x'$$
$$= -\frac{\alpha}{2}\int_\Sigma \frac{\partial}{\partial t}\frac{|\nabla'\eta_{x_i}|^2}{\sqrt{1 + |\nabla'\eta|^2}}\,\mathrm{d}x' - \frac{\alpha}{2}\int_\Sigma \frac{|\nabla'\eta_{x_i}|^2\nabla'\eta \cdot \nabla'\eta_t}{\sqrt{(1 + |\nabla'\eta|^2)^3}}\,\mathrm{d}x' + \alpha\int_\Sigma \frac{\nabla'\eta \cdot \nabla'\eta_{x_i}\nabla'\eta \cdot \nabla'\eta_{tx_i}}{\sqrt{(1 + |\nabla'\eta|^2)^3}}\,\mathrm{d}x'$$
$$= -\frac{\alpha}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Sigma \frac{|\nabla'\eta_{x_i}|^2}{\sqrt{1 + |\nabla'\eta|^2}}\mathrm{d}x' + \frac{\alpha}{2}\int_\Sigma \frac{\frac{\partial}{\partial t}((\nabla'\eta \cdot \nabla'\eta_{x_i})^2) - |\nabla'\eta_{x_i}|^2\nabla'\eta \cdot \nabla'\eta_t - 2\nabla'\eta \cdot \nabla'\eta_{x_i}\nabla'\eta_{x_i} \cdot \nabla'\eta_t}{\sqrt{(1 + |\nabla'\eta|^2)^3}}\,\mathrm{d}x'$$

$$= -\frac{\alpha}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Sigma} \frac{|\nabla'\eta_{x_i}|^2(1+|\nabla'\eta|^2) - (\nabla'\eta\cdot\nabla'\eta_{x_i})^2}{\sqrt{(1+|\nabla'\eta|^2)^3}}\,\mathrm{d}x' + B_{14}^{(i)}$$

$$= -\frac{\alpha}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Sigma} \frac{|\nabla'\eta_{x_i}|^2 + (\eta_{x_1}\eta_{x_i x_2} - \eta_{x_2}\eta_{x_i x_1})^2}{\sqrt{(1+|\nabla'\eta|^2)^3}} + B_{14}^{(i)} = -\frac{\alpha}{2}\frac{\mathrm{d}}{\mathrm{d}t}K^{(x_i)} + B_{14}^{(i)},$$

where

$$B_{14}^{(i)} = \frac{\alpha}{2}\int_{\Sigma} \frac{-|\nabla'\eta_{x_i}|^2\nabla'\eta\cdot\nabla'\eta_t - 2\nabla'\eta\cdot\nabla'\eta_{x_i}\nabla'\eta_{x_i}\cdot\nabla'\eta_t}{\sqrt{(1+|\nabla'\eta|^2)^3}}\,\mathrm{d}x' + \frac{3\alpha}{2}\int_{\Sigma} \frac{(\nabla'\eta\cdot\nabla'\eta_{x_i})^2\nabla'\eta\cdot\nabla'\eta_t}{\sqrt{(1+|\nabla'\eta|^2)^5}}\,\mathrm{d}x'. \tag{A.6}$$

In order to obtain the dissipative term corresponding to the perturbation η on the height of the layer, we use the idea of introducing a generalized energy [3,5] and the auxiliary vector field constructed in [3]. It is proved in [3] that there exists a vector field $\mathbf{V} \in L^{\infty}(0,\infty; W_{\natural}^{1,2}(\Omega_t))$ with $\partial\mathbf{V}/\partial t \in L^{\infty}(0,\infty; L_{\natural}^{2}(\Omega))$ which satisfies the following conditions

$$\nabla\cdot\mathbf{V} = 0, \quad x\in\Omega_t,$$

$$\mathbf{V}(x',0,t) = 0,$$

$$\mathbf{V}\cdot\mathbf{n}|_{\Gamma_t} = \eta n_3,$$

and the following estimates hold true:

$$\left\|\frac{\partial\mathbf{V}}{\partial t}\right\|_{L_2(\Omega)} \leqslant c\left\|\frac{\partial\eta}{\partial t}\right\|_{L_2(\Sigma)}, \qquad \|\nabla\mathbf{V}\|_{L_2(\Omega)} \leqslant c\|\eta\|_{W^{1,2}(\Sigma)}.$$

We multiply Eq. $(\mathrm{A}.1)_1$ by the auxiliary vector field $\mathbf{V}$ differentiated with respect to x_i, integrate over Ω_t, and integrate by parts, we obtain

$$\int_{\Omega_t}\rho\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{u}_{x_i}\cdot\mathbf{V}_{x_i})\,\mathrm{d}x - \int_{\Omega_t}\rho\mathbf{u}_{x_i}\cdot\frac{\mathrm{d}\mathbf{V}_{x_i}}{\mathrm{d}t}\,\mathrm{d}x + \int_{\Omega_t}\sigma_{x_i}\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t}\cdot\mathbf{V}_{x_i}\,\mathrm{d}x + \int_{\Omega_t}\rho(\mathbf{u}_{x_i}\cdot\nabla)\mathbf{u}\cdot\mathbf{V}_{x_i}\,\mathrm{d}x + \mu\big(\mathbf{D}(\mathbf{u}_{x_i}),\nabla\mathbf{V}_{x_i}\big)$$
$$= \int_{\Gamma_t}\mathbf{V}_{x_i}\cdot\big(\mu\mathbf{D}(\mathbf{u}_{x_i}) - \mathbf{I}p_{x_i}\big)\mathbf{n}\,\mathrm{d}s - g\int_{\Omega_t}\sigma_{x_i}\nabla x_3\cdot\mathbf{V}_{x_i}\,\mathrm{d}x. \tag{A.7}$$

Differentiating the boundary condition $\mathbf{V}\cdot\mathbf{n}^{\wedge}|_{\Gamma_t} = \eta$ with respect to x_i, we have

$$\mathbf{V}_{x_i}\cdot\mathbf{n}^{\wedge} = \eta_{x_i} - \mathbf{V}\cdot\mathbf{n}^{\wedge}_{x_i} - \mathbf{V}_{x_3}\cdot\mathbf{n}^{\wedge}\zeta_{x_i},$$

so, with the help of relation (A.4), we can write the boundary term in the right-hand side of (A.7) in the form

$$\int_{\Gamma_t}\mathbf{V}_{x_i}\cdot\big(\mu\mathbf{D}(\mathbf{u}_{x_i}) - \mathbf{I}p_{x_i}\big)\mathbf{n}\,\mathrm{d}s = -\alpha\int_{\Sigma}\frac{|\nabla'\eta_{x_i}|^2 + (\eta_{x_1}\eta_{x_i x_2} - \eta_{x_2}\eta_{x_i x_1})^2}{\sqrt{1+|\nabla'\eta|^2}}\,\mathrm{d}x' + D^{(i)},$$

where

$$D^{(i)} = -\alpha\int_{\Sigma}\frac{\partial\mathcal{H}}{\partial x_i}\big(\mathbf{V}\cdot\mathbf{n}^{\wedge}_{x_i} + \mathbf{V}_{x_3}\cdot\mathbf{n}^{\wedge}\zeta_{x_i}\big)\,\mathrm{d}x' + \int_{\Gamma_t}p_{x_3}\zeta_{x_i}\mathbf{n}\cdot\mathbf{V}_{x_i}\,\mathrm{d}s$$
$$- \mu\int_{\Gamma_t}\mathbf{V}_{x_i}\cdot\mathbf{D}(\mathbf{u}_{x_3})\mathbf{n}\zeta_{x_i}\,\mathrm{d}s + \mu\int_{\Gamma_t}\big(\mathbf{V}_{x_i} - (\mathbf{n}\cdot\mathbf{V}_{x_i})\mathbf{n}\big)\cdot\mathbf{D}(\mathbf{u}_{x_i})\mathbf{n}\,\mathrm{d}s. \tag{A.8}$$

We multiply (A.7) by the small number $\gamma_i > 0$ and add to (A.2), taking into account the calculations for the boundary terms which is done above, we have:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_t}\rho|\mathbf{u}_{x_i}|^2\,\mathrm{d}x + \frac{\alpha}{2}\frac{\mathrm{d}}{\mathrm{d}t}K^{(x_i)} + \gamma_i\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_t}\rho\mathbf{u}_{x_i}\cdot\mathbf{V}_{x_i}\,\mathrm{d}x + 2\mu\big\|\mathbf{D}(\mathbf{u}_{x_i})\big\|^2 + \gamma_i\alpha K^{(x_i)} + 2\mu\gamma_i\big(\mathbf{D}(\mathbf{u}_{x_i}), D(\mathbf{V}_{x_i})\big)$$
$$= P^{(x_i)} + R^{(i)} + B_{12}^{(i)} + B_{13}^{(i)} + B_{14}^{(i)} + B_2^{(i)} + \gamma_i D^{(i)}, \tag{A.9}$$

where

$$
\begin{aligned}
&R^{(i)} = -\int_{\Omega_t} \rho(\mathbf{u}_{x_i} \cdot \nabla)\mathbf{u} \cdot (\mathbf{u}_{x_i} + \gamma_i \mathbf{V}_{x_i})\,\mathrm{d}x + \gamma_i \int_{\Omega_t} \rho \mathbf{u}_{x_i} \cdot \frac{\mathrm{d}\mathbf{V}_{x_i}}{\mathrm{d}t}\,\mathrm{d}x,\\
&B_{12}^{(i)} = -\alpha \int_{\Sigma} \frac{\partial \mathcal{H}}{\partial x_i} \mathbf{u} \cdot \mathbf{n}_{x_i}^{\wedge}\,\mathrm{d}x', \qquad B_{13}^{(i)} = -\alpha \int_{\Sigma} \frac{\partial \mathcal{H}}{\partial x_i} \mathbf{u}_{x_3} \cdot \mathbf{n}^{\wedge} \zeta_{x_i}\,\mathrm{d}x',
\end{aligned}
$$

the term $B_{14}^{(i)}$ is given by (A.6), $B_2^{(i)}$ are given by (A.5), $D^{(i)}$ are given by (A.8), and $K^{(x_i)}$ by (4.2).

Let us take a derivative with respect to t of all the equations in problem (3.1), we have:

$$
\begin{aligned}
&\sigma_t \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} + \rho\left(\frac{\mathrm{d}\mathbf{u}_t}{\mathrm{d}t} + (\mathbf{u}_t \cdot \nabla)\mathbf{u}\right) - \mu \Delta \mathbf{u}_t = \nabla p_t - \sigma_t g \nabla x_3, \quad x \in \Omega_t,\\
&\nabla \cdot \mathbf{u}_t = 0, \quad x \in \Omega_t,\\
&\frac{\mathrm{d}\sigma_t}{\mathrm{d}t} + \mathbf{u}_t \cdot \nabla(\rho_b + \sigma) = 0, \quad x \in \Omega_t.
\end{aligned}
\tag{A.10}
$$

We multiply $(\mathrm{A}.10)_1$ by $\mathbf{u}_t$, integrate over Ω_t, and arrive at

$$
\begin{aligned}
&\int_{\Omega_t} \sigma_t \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} \cdot \mathbf{u}_t\,\mathrm{d}x + \int_{\Omega_t} \rho \frac{\mathrm{d}\mathbf{u}_t}{\mathrm{d}t} \cdot \mathbf{u}_t\,\mathrm{d}x + \int_{\Omega_t} \rho(\mathbf{u}_t \cdot \nabla)\mathbf{u} \cdot \mathbf{u}_t\,\mathrm{d}x - \mu \int_{\Omega_t} \Delta \mathbf{u}_t \cdot \mathbf{u}_t\,\mathrm{d}x\\
&\quad = -\int_{\Omega_t} \nabla p \cdot \mathbf{u}_t\,\mathrm{d}x - g \int_{\Omega_t} \sigma_t \nabla x_3 \cdot \mathbf{u}_t\,\mathrm{d}x.
\end{aligned}
\tag{A.11}
$$

Because of $(\mathrm{A}.10)_2$, with the help of integration by parts, we obtain

$$
\int_{\Omega_t} (\mu \Delta \mathbf{u}_t - \nabla p_t) \cdot \mathbf{u}_t\,\mathrm{d}x = -2\mu \int_{\Omega_t} \mathbf{D}(\mathbf{u}_t) : \mathbf{D}(\mathbf{u}_t)\,\mathrm{d}x + \int_{\Gamma_t} \mathbf{u}_t \cdot \big(-p_t \mathbf{n} + 2\mu \mathbf{D}(\mathbf{u}_t)\mathbf{n}\big)\,\mathrm{d}s.
\tag{A.12}
$$

We take a derivative with respect to t of the boundary condition $(3.1)_5$, and have

$$
2\mu \mathbf{D}(\mathbf{u})\mathbf{n}_t + 2\mu \mathbf{D}(\mathbf{u}_t)\mathbf{n} + 2\mu \mathbf{D}(\mathbf{u}_{x_3})\mathbf{n}\zeta_t = (\alpha \mathcal{H} - p_e + p)\mathbf{n}_t + \left(\alpha \frac{\partial \mathcal{H}}{\partial t} + p_t + p_{x_3}\zeta_t\right)\mathbf{n}.
$$

Because of the fact that $\mathbf{n}_t \cdot \mathbf{n} = 0$, it implies

$$
p_t = 2\mu \mathbf{n} \cdot \mathbf{D}(\mathbf{u}_{x_3})\mathbf{n}\zeta_t - p_{x_3}\zeta_t - \alpha \frac{\partial \mathcal{H}}{\partial t} + 2\mu \mathbf{n} \cdot \mathbf{D}(\mathbf{u}_t)\mathbf{n}.
\tag{A.13}
$$

Taking a derivative with respect to t of the boundary condition $(3.1)_6$, we obtain

$$
\mathbf{u}_t \cdot \mathbf{n}^{\wedge} = \eta_{tt} - \mathbf{u} \cdot \mathbf{n}_t^{\wedge} - \mathbf{u}_{x_3} \cdot \mathbf{n}^{\wedge} \zeta_t.
\tag{A.14}
$$

We substitute (A.13), (A.14) in the boundary term obtained in (A.12) and apply the same procedure as above for the term $B_1^{(i)}$, this gives us the following relation

$$
\begin{aligned}
&\int_{\Omega_t} (\mu \Delta \mathbf{u}_t - \nabla p_t) \cdot \mathbf{u}_t\,\mathrm{d}x\\
&\quad = -2\mu \big\|\mathbf{D}(\mathbf{u}_t)\big\|^2 - \frac{\alpha}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Sigma} \frac{|\nabla' \eta_t|^2 + (\eta_{x_1}\eta_{tx_2} - \eta_{x_2}\eta_{tx_1})^2}{\sqrt{(1 + |\nabla' \eta|^2)^3}}\,\mathrm{d}x' + B_2^{(t)} + B_{14}^{(t)} + B_{12}^{(t)} + B_{13}^{(t)},
\end{aligned}
\tag{A.15}
$$

where

$$
\begin{aligned}
&B_2^{(t)} = \int_{\Gamma_t} p_{x_3}\zeta_t \mathbf{n} \cdot \mathbf{u}_t\,\mathrm{d}s - 2\mu \int_{\Gamma_t} \mathbf{u}_t \cdot \mathbf{n}\mathbf{n} \cdot \mathbf{D}(\mathbf{u}_{x_3})\mathbf{n}\zeta_t\,\mathrm{d}s + 2\mu \int_{\Gamma_t} \big(\mathbf{u}_t - (\mathbf{n} \cdot \mathbf{u}_t)\mathbf{n}\big) \cdot \mathbf{D}(\mathbf{u}_t)\mathbf{n}\,\mathrm{d}s,\\
&B_{14}^{(t)} = \frac{3\alpha}{2} \int_{\Sigma} \frac{(\nabla' \eta \cdot \nabla' \eta_t)^3 - |\nabla' \eta_t|^2 (1 + |\nabla' \eta|^2) \nabla' \eta \cdot \nabla' \eta_t}{\sqrt{(1 + |\nabla' \eta|^2)^5}}\,\mathrm{d}x',\\
&B_{12}^{(t)} = -\alpha \int_{\Sigma} \frac{\partial \mathcal{H}}{\partial t} \mathbf{u} \cdot \mathbf{n}_t^{\wedge}\,\mathrm{d}x', \qquad B_{13}^{(t)} = -\alpha \int_{\Sigma} \frac{\partial \mathcal{H}}{\partial t} \mathbf{u}_{x_3} \cdot \mathbf{n}^{\wedge} \zeta_t\,\mathrm{d}x'.
\end{aligned}
\tag{A.16}
$$

To obtain the dissipative term with the L_2 norm of $|\nabla'\eta_t|$, we multiply Eq. $(A.10)_1$ by the auxiliary vector field $\mathbf{V}$ differentiated with respect to t, integrate over Ω_t, and use the same scheme as above for the derivatives with respect to x_i. As a consequence, taking into account (A.11), (A.15), we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\Omega_t}\rho|\mathbf{u}_t|^2\,\mathrm{d}x+\gamma_0\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\Omega_t}\rho\mathbf{u}_t\cdot\mathbf{V}_t\,\mathrm{d}x+\frac{\alpha}{2}\frac{\mathrm{d}}{\mathrm{d}t}K^{(t)}+\alpha\gamma_0K^{(t)}+2\mu\big\|\mathbf{D}(\mathbf{u}_t)\big\|^2+2\mu\gamma_0\big(\mathbf{D}(\mathbf{u}_t),D(\mathbf{V}_t)\big)$$
$$=P^{(t)}+R^{(t)}+B_2^{(t)}+B_{14}^{(t)}+B_{12}^{(t)}+B_{13}^{(t)}+\gamma_0D^{(t)}, \tag{A.17}$$

where

$$R^{(t)}=-\int\limits_{\Omega_t}\rho(\mathbf{u}_t\cdot\nabla)\mathbf{u}\cdot(\mathbf{u}_t+\gamma_0\mathbf{V}_t)\,\mathrm{d}x+\gamma_0\int\limits_{\Omega_t}\rho\mathbf{u}_t\cdot\frac{\mathrm{d}\mathbf{V}_t}{\mathrm{d}t}\,\mathrm{d}x,\quad \gamma_0>0,$$

the terms $B_k^{(t)}$ are given by (A.16), and $K^{(t)}$ by (4.2),

$$D^{(t)}=-\alpha\int\limits_{\Sigma}\frac{\partial\mathcal{H}}{\partial t}\big(\mathbf{V}\cdot\mathbf{n}_t^{\wedge}+\mathbf{V}_{x_3}\cdot\mathbf{n}^{\wedge}\zeta_t\big)\,\mathrm{d}x'+\int\limits_{\Gamma_t}p_{x_3}\zeta_t\mathbf{n}\cdot\mathbf{V}_t\,\mathrm{d}s-2\mu\int\limits_{\Gamma_t}\mathbf{V}_t\cdot\mathbf{D}(\mathbf{u}_{x_3})\mathbf{n}\zeta_t\,\mathrm{d}s$$
$$+2\mu\int\limits_{\Gamma_t}\big(\mathbf{V}_t-(\mathbf{n}\cdot\mathbf{V}_t)\mathbf{n}\big)\cdot\mathbf{D}(\mathbf{u}_t)\mathbf{n}\,\mathrm{d}s.$$

Taking a sum of (A.9), (A.17) we arrive at (4.1), with

$$N(\mathbf{u},\eta)=R^{(1)}+R^{(2)}+R^{(t)}+\gamma_1D^{(1)}+\gamma_2D^{(2)}+\gamma_0D^{(t)}+B_2^{(1)}+B_2^{(2)}+B_2^{(t)}$$
$$+B_{12}^{(1)}+B_{12}^{(2)}+B_{12}^{(t)}+B_{13}^{(1)}+B_{13}^{(2)}+B_{13}^{(t)}+B_{14}^{(1)}+B_{14}^{(2)}+B_{14}^{(t)},$$

which completes the proof of lemma. □

References

[1] B. Helffer O. Lafitte, Asymptotic methods for the eigenvalues of the Rayleigh equation for linearized Rayleigh–Taylor instability, Preprint Univ. Paris-Sud Math. 2424, 2002, pp. 1–49.
[2] C. Cherfils-Clerouin, O. Lafitte, Analytic solutions of the Rayleigh equation for linear density profiles, Phys. Rev. E 62 (2000) 2967–2970.
[3] M. Padula, V.A. Solonnikov, On Rayleigh–Taylor stability, Ann. Univ. Ferrara Sez. VII 46 (2000) 307–336.
[4] B.J. Jin, M. Padula, On the existence of compressible viscous flow in a horizontal layer with free upper surface, Comm. Pure Appl. Anal. 1 (3) (2002) 370–415.
[5] M. Padula, On the exponential decay to the rest state for a viscous isothermal fluid, J. Fluid Mech. Anal. 1 (1998).
[6] M. Padula, On direct Ljapunov method in continuum theories, in: Sh. Birman, S. Hildebrandt, V.A. Solonnikov, N.N. Uralsteva (Eds.), Nonlinear Problems in Physics and Related Topics I, in honor of prof. O.A. Ladyzhenskaya, Novosibirsk, 2002, pp. 271–283, Kluwer Academic/Plenum, New York, 2002, pp. 289–302.
[7] A.V. Kajikov, Resolution of boundary value problems for nonhomogeneous viscous fluids, Dokl. Akad. Nauk 216 (5) (1974) 1008–1010.
[8] S.N. Antonsev, A.V. Kajikov, The mathematical problems of the dynamics of nonhomogeneous fluids, Lectures of the University, Novosibirsk, 1973.
[9] O.A. Ladyzhenskaya, V.A. Solonnikov, On unique solvability of an initial-boundary value problem for viscous incompressible nonhomogeneous fluids, J. Soviet Math. 9 (1978) 697–749.
[10] M. Padula, On existence theorem for nonhomogeneous incompressible fluids, Rend. Circ. Mat. Palermo (2) Suppl. 31 (1982) 119–124.
[11] A.A. Arkhipova, O.A. Ladyzhenskaya, On inhomogeneous incompressible fluids and reverse Holder inequalities, Ann. Scuola Norm. Sup. Pisa Cl. Sci (4) 25 (1997) 51–67.
[12] J. Simon, Sur les fluides visqueux incompressibles et non-homogènes, C. R. Acad. Sci. Paris, Ser. I 309 (1989) 447–452.
[13] J.-L. Lions, On Some Problems Connected with Navier–Stokes Equations, Nonlinear Evolution Equation, Academic Press, 1978, pp. 59–84.
[14] J.T. Beale, The initial value problem for the Navier–Stokes equations with a free surface, Comm. Pure Appl. Math. 34 (1981) 359–392.